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# KAEHLER LIGHTLIKE SUBMANIFOLDS

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## Abstract

In this paper, we introduce a class of lightlike submanifolds of Kaehler manifold. We show that the induced connection is metric. We also endow theirs with symplectic form. Finally, we obtain a class of lightlike symplectic submanifolds. Hamiltonian formulation and symplectic reduction machinery in lightlike are presented.

## 1. Introduction

The growing importance of lightlike submanifolds in global Lorentzian geometry and their application in general relativity motivated the study of degenerate manifolds. Recently, that is during two to three last decades, the use of general null geometry theory as a mathematical foundation in the study of massless objects in physics has become very important [11, 12]. This study is becoming one of interesting topics in differential geometry of submanifolds of semi-Riemannian manifolds.

It is also known that classical mechanics, in its Hamiltonian formulation on the motion space, has for framework a symplectic manifold. Smooth functions on that manifold are observable and the

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dynamics is defined in terms of Hamiltonian $H$ and time evolution of an observable $f_t$ is governed by the equation

$$\frac{d}{dt} f_t = -\{H, f_t\}.$$

For this paper, we want to make link between the lightlike geometry and classical mechanics in its Hamiltonian formulation. It is possible to have, in this direction, the results who can explain or develop more one of these two theory.

Our aim in this paper is to give a class of Kaehler lightlike submanifolds of an indefinite Kaehler manifold, which is invariant under the complex structure. There exist many works on CR-lightlike manifolds. But, in this work, the radical distribution is invariant under a complex structure. In [7], we introduced a volume element of lightlike hypersurface. We give a generalisation of this volume element on a lightlike submanifold. In Riemannian manifolds, it is known that the Kaehler form is a symplectic, even if Thurston [13] showed that the converse is not true and give some examples. But, it is come evident that in the complex lightlike submanifold of an indefinite Kaehler manifold, the Kaehler form is not symplectic contrary to pseudo-Riemannian manifold case.

Moreover, if $f : M \to (\overline{M}, \overline{g}, \overline{J}, w)$ is holomorphic isometric immersion of complex submanifold $M$ into pseudo symplectic manifold $(\overline{M}, \overline{g}, \overline{J}, w)$, then $f^\star w$ is symplectic form on $M$ [4, pg.76]. This result is not true in the lightlike one. We show that induced connection on $M$ is metric.

**Theorem 1.1.** *Let* $f : (M^{2n}, g, S(TM), S(TM^\perp)) \to (\overline{M}^{2m}, \overline{g}, \overline{J}, w)$ *be a holomorphic isometric immersion of lightlike Kaehler submanifold* $M^{2n}$ *into an indefinite Kaehler manifolds* $\overline{M}^{2m}$. *Then an induced connection on M is metric. Moreover,* $f^\star w$ *is no symplectic.*

**Corollary 1.1.** *Let* $f : (M^{2n}, g, S(TM)) \to (\overline{M}^{2m}, \overline{g}, \overline{J}, w)$ *be a holomorphic isometric immersion of a lightlike coisotrope almost complex submanifold into an indefinite Kaehler manifolds. Then* $(M^{2n}, g)$ *is totally geodesic.*

This corollary generalize the result obtained by Fazilet in [8, Proposition 7.3]. This paper is only which talk about this subject, but in small section without geometry details but in Kupeli approach [10].

As another consequence, a lightlike almost complex submanifold $M$ is totally geodesic or minimal.

**Corollary 1.2.** *Let* $f : (M^{2n}, g, S(TM), S(TM^{\perp})) \to (\overline{M}^{2m}, \overline{g}, \overline{J}, w)$ *be a holomorphic isometric immersion of a lightlike complex submanifold* $M^{2n}$ *into an indefinite Kaehler manifolds* $\overline{M}^{2m}$. *Then* $M^{2n}$ *is minimal or totally geodesic.*

We use the lightlike metric to endow this class of submanifolds with symplectic form. Thus, for this normalization, we obtain necessary condition for lightlike vector fields to be Hamiltonian. The symplectic reduction machinery is also used.

**Theorem 1.2.** *Let* $(M^{2n}, S(TM), S(TM^{\perp}), g, J)$ *be an invariant holomorphic* 2*-lightlike almost complex submanifolds of an indefinite symplectic manifolds* $(\overline{M}^{2m}, \overline{g}, \overline{J}, w)$ *and normalizing null vectors given in Lemma* 4.1. *Then an associate* 2*-form* $\eta = f^{\star}w + \theta \wedge J\theta$ *is a symplectic form on M.*

**Theorem 1.3.** *Let* $(M, g, S(TM), S(TM^{\perp}), \eta)$ *be a* 2*-lightlike invariant submanifold endowed with associate symplectic form* $\eta$. *Then* $Rad(TM) \subset \mathfrak{L}(M, \eta)$.

**Theorem 1.4.** *If the screen leaf space* $M / \mathfrak{F} = P$ *is a smooth manifold, then* $(P, (f^{\star}w)_{\setminus P})$ *is Kaehler* (*symplectic manifold*).

The paper is organized as follows: In Section 2, we give preliminaries which come essentially to Duggal and Bejancu [5] for lightlike submanifolds and to Cannas [4] for Kaehler manifolds. We also give, in this section, a generalization of lightlike submanifolds metric volume.

In Section 3, we present an invariant lightlike Kaehler submanifold. We prove the Theorem 1.1 and Corollaries 1.1, 1.2. We end this section by examples. In Section 4, we use then another normalization to associate the symplectic form to lightlike Kaehler submanifolds. We prove Theorem 1.2. In Subsection 4.3, the proof of Theorem 1.3 and lightlike Hamiltonian vector fields are presented. The last Section 5, in which the proof of Theorem 1.4 deals with symplectic reduction machinery in lightlike Kaehler submanifolds.

## 2. Preliminaries

### 2.1. Basics on lightlike submanifolds

Let $(M^{2n}, g)$ be an $r$-lightlike submanifolds of pseudo-Riemannian manifold $(\overline{M}^{2m}, \overline{g})$. Then one has $(2p)$-dimensional vector space $(p = m - n)$.

$$T_x M^{\perp} = \{v_x \in T_x \overline{M}, \overline{g}(v_x, u_x) = 0, \forall u_x \in T_x M\},$$

and

$$RadTM = TM \cap TM^{\perp} \neq \{0\}$$

is $r$-dimensional subspace. There exists four kinds of lightlike submanifolds:

- The proper $r$-lightlike submanifolds, where $0 < r < \min(2n, 2p)$. In this case, $Rad(TM) \subsetneq TM$ and $Rad(TM) \subsetneq TM^{\perp}$.

- The coisotropic submanifolds, when $1 < r = 2p < 2n$. Then, $Rad(TM) = TM^{\perp} \subsetneq TM$.

• The isotropic submanifolds case, when $1 < r = 2n < 2p$. Then, $Rad(TM) = TM \subsetneq TM^{\perp}$.

• The totally lightlike submanifolds, when $1 < r = 2p = 2n$. Then, $Rad(TM) = TM = TM^{\perp}$.

In the first case, $T\overline{M}$ has a follow decomposition:

$$\begin{aligned} T\overline{M}_{|M} &= TM \oplus tr(TM) \\ &= (TM \oplus ltr(TM)) \perp S(TM^{\perp}), \end{aligned} \tag{1}$$

and

$$TM = Rad(TM) \perp S(TM), \tag{2}$$

$$TM^{\perp} = Rad(TM) \perp S(TM^{\perp}), \tag{3}$$

where $S(TM)$, $S(TM^{\perp})$ are screen distribution, screen transversal vector bundle, which is a complementary vector bundle of $Rad(TM)$ in $TM^{\perp}$ and $ltr(TM)$ is a lightlike transversal vector bundle. For any local basis $\{\xi_i\}$ of $Rad(TM)$, there exists a local frame $\{N_i\}$ of $ltr(TM)$ such that $\overline{g}(\xi_i, N_i) = \delta_{ij}$, $\overline{g}(N_j, N_i) = 0$, and $\overline{g}(W_j, N_i) = 0$ for any local frame $\{W_i\}$ of $S(TM)$. In the coisotrope case, $S(TM^{\perp}) = \{0\}$. Then the relation (1) becomes

$$T\overline{M}_{|M} = TM \oplus ltr(TM). \tag{4}$$

In the third case, the isotropic submanifold gives $S(TM) = \{0\}$ and

$$T\overline{M}_{|M} = (Rad(TM) \oplus ltr(TM)) \perp S(TM^{\perp}). \tag{5}$$

In the last case, $S(TM) = S(TM^{\perp}) = \{0\}$.

Let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$ and $\nabla$ be induced Levi-Civita connection on $M$. Then according to relations (1) and (4);

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall\ X, Y \in \Gamma(TM), \tag{6}$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \quad \forall X \in \Gamma(TM), \quad V \in \Gamma(tr(TM)), \tag{7}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X, Y), \nabla^t_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively.

Suppose that $S(TM^{\perp}) \neq \{0\}$ and consider the projection morphisms $L$ and $S$ of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively.

$$L : tr(TM) \to ltr(TM),$$

$$S : tr(TM) \to S(TM^{\perp}).$$

Then, the relations (6) and (7) become

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \forall\ X, Y \in \Gamma(TM), \tag{8}$$

where $h^l(X, Y) = L(h(X, Y))$ and $h^s(X, Y) = S(h(X, Y))$.

$$\overline{\nabla}_X V = -A_V X + D^l_X V + D^s_X V, \quad \forall\ X \in \Gamma(TM), \quad V \in \Gamma(tr(TM)), \tag{9}$$

where $D^l_X V = L(\nabla^t_X V)$ and $D^s_X V = S(\nabla^t_X V)$.

For any $X \in \Gamma(TM)$,

$$\nabla^l_X : \Gamma(ltr(TM)) \to \Gamma(ltr(TM)), \quad \nabla^l_X(LV) = D^l_X(LV), \tag{10}$$

and

$$\nabla^s_X : \Gamma(S(TM^{\perp})) \to \Gamma(S(TM^{\perp})), \quad \nabla^s_X(SV) = D^s_X(SV), \tag{11}$$

for any $V \in \Gamma(tr(TM))$. Then, we define

$$D^l : \Gamma(TM) \times \Gamma(S(TM^{\perp})) \to \Gamma(ltr(TM)), \quad D^l(X, SV) = D^l_X(SV), \tag{12}$$

$$D^s : \Gamma(TM) \times \Gamma(ltr(TM)) \to \Gamma(S(TM^\perp)), \quad D^s(X, LV) = D^s_X(LV), \quad (13)$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$.

Thus, the relation (9) becomes

$$\overline{\nabla}_X V = -A_V X + D^l(X, SV) + D^s(X, LV) + \nabla^l_X(LV) + \nabla^s_X(SV). \quad (14)$$

The above different geometry objects verify the following relations:

$$\overline{g}(h^s(X, Y), W) + \overline{g}(Y, D^l(X, W)) = \overline{g}(A_W X, Y), \quad (15)$$

$$\overline{g}(h^l(X, Y), \xi) + \overline{g}(Y, h^l(X, \xi)) + \overline{g}(Y, \nabla_X \xi) = 0, \quad (16)$$

$$\overline{g}(W, D^s(X, N)) = \overline{g}(A_W X, N), \quad (17)$$

$$\overline{g}(A_N X, N') = \overline{g}(A_{N'} X, Y), \quad (18)$$

$$\overline{g}(A_N X, PY) = \overline{g}(N, \nabla_X PY), \quad (19)$$

$$h^l_i(X, \xi_j) = h^l_j(X, \xi_i), \quad (20)$$

where $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$, $\xi_i \in \Gamma(Rad(TM))$, $W \in \Gamma(S(TM^\perp))$, and $h_i$ are such that

$$h_i(X, Y) = g(\overline{\nabla}_X Y, \xi_i), \quad \forall X, Y \in \Gamma(TM). \quad (21)$$

In general, the induced connection $\nabla$ on $M$ and transversal connection $\nabla^t$ on $tr(TM)$ are not metric. Thus,

$$(\nabla_X g)(Y, Z) = \overline{g}(h^l(X, Y), Z) + \overline{g}(h^l(X, Z), Y), \quad (22)$$

and

$$\nabla^t_X(\overline{g})(V, V') = -(\overline{g}(A_V X, V') + \overline{g}(A_{V'} X, V)). \quad (23)$$

**2.2. Basics on Kaehler manifolds**

Let $(\overline{M}^{2m}, \overline{g})$ be a pseudo-Riemannian manifold and suppose there exists an automorphism $\overline{J}$ on $T\overline{M}$ such that $\overline{J}^2 = -\mathbb{I}$, where $\mathbb{I}$ is the identity on $T\overline{M}$ and

$$\overline{g}(\overline{J}_x X, \overline{J}_x Y) = \overline{g}(X, Y); \quad \forall X, Y \in T_x M.$$

Recall that the index of such metric is even. Then $(\overline{M}, \overline{g}, \overline{J})$ is called an *indefinite Hermitian manifold*, if the Nijenhuis tensor fields of $\overline{J}$ vanished $(N_{\overline{J}} = 0)$, where

$$N_{\overline{J}}(X, Y) = [\overline{J}X, \overline{J}Y] - [X, Y] - \overline{J}([X, \overline{J}Y] + [\overline{J}X, Y]). \tag{24}$$

Moreover, according to Barros and Romero [3], $\overline{M}$ is called an *indefinite Kaehler manifolds*, if $\overline{J}$ is parallel with respect to $\overline{\nabla}$, the Levi-Civita connection of $(\overline{M}, g)$. That is,

$$\overline{\nabla}\overline{J} = 0. \tag{25}$$

The existence of complex structure $\overline{J}$ is not always guaranteed. As the 6-dimensional sphere $S^6$ has no complex structure. However, $S^6$ carries an almost complex structure [9].

Let $w$ be a 2-form on $\overline{M}$ define as follows:

$$w(X, Y) = \overline{g}(X, \overline{J}Y), \quad \forall X, Y \in \Gamma(T\overline{M}). \tag{26}$$

Then $w$ is called *Kaehler form* and define a symplectic form on $\overline{M}$. Moreover, $(\overline{M}, \overline{g}, \overline{J}, w)$ is an indefinite symplectic manifolds. But, it is shown by Thurston [13] that a symplectic manifold not always Kaehler. It is also comment by Banyaga [2].

Let $M$ be a submanifold of a symplectic manifold $(\overline{M}, w)$. We note

$$T_x M^{\perp_w} = \{v_x \in T_x M, \quad w_x(v_x, u_x) = 0, \quad \forall u_x \in T_x M\},$$

and

$$Rad^w(TM) = TM \cap TM^{\perp_w}.$$

**2.3. Metric volume element on lightlike submanifolds**

Let $(M, g, S(TM), S(TM^{\perp}))$ be an $m$-dimensional lightlike submanifolds of an oriented $(m + p)$-dimensional semi-Riemannian $(\overline{M}, \overline{g})$ and $i : M \to \overline{M}$ be an isometric immersion.

Suppose $s : \overline{M} \to \mathbb{R}^p$ is a submersion such that there exists $a \in \mathbb{R}^p$, which gives $s^{-1}(a) = M$.

**Proposition 2.1.** *The p-form* $w = ds_1 \wedge ds_2 \wedge ds_3 \wedge \ldots \wedge ds_p$ *is nonull everywhere, where* $s = (s_1, \ldots, s_p)$. *Moreover, there exists on* $\overline{M}$ *the m-form* $\eta$ *such that*

$$(w \wedge \eta)(x) = v_{\overline{M}}(x), \quad x \in M, \tag{27}$$

*where* $v_{\overline{M}}$ *is an element of volume on* $\overline{M}$.

*Then, the volume form* $v_M$ *on M is characterized by*

$$v_M(x) = i^*(\eta)(x).$$

**Proof 2.1.** $w$ is everywhere nonull. Indeed, $s$ is a submersion and the rank of the Jacobian of $s$ is constant and equal to $p$ on $M$.

(1) Existence of $\eta$ :

We can take $\eta$ as

$$\eta = v_{\overline{M}}((ds_1)^{\#}, \ldots, (ds_p)^{\#}),$$

where $(ds_i)^{\#}$ is a vector field such that $ds_i((ds_j)^{\#}) = \delta_{ij}$.

This vector field $(ds_i)^{\#}$ is easily obtained because of nodegenerate of the metric $\overline{g}$.

(2) Uniqueness of $\upsilon_M$ :

Suppose there exists $\eta$ and $\eta'$ such that

$$w \wedge \eta = w \wedge \eta' = \upsilon_{\overline{M}},$$

then

$$w \wedge (\eta - \eta') = 0, \quad \eta - \eta' = W \wedge \gamma,$$

where $\gamma$ is a $(m - p)$-form on $M$.

Let $X$ be the vector field on $M$, then $w(X) = 0.$ Thus,

$$i^*\eta = i^*\eta' = \upsilon_M.$$

**Remark 2.1.** This volume element is independent to the choice of screen distribution. The orientation of lightlike submanifolds comes the orientation of $\overline{M}$ as follows. Let $w = ds_1 \wedge ds_2 \wedge ds_3 \wedge \ldots \wedge ds_p,$ we say that $M$ has direct orientation, if $\det(w \wedge \eta) > 0.$ The indirect is given by $\det(w \wedge \eta) < 0.$ In another way, let $f : M^p \to \overline{M}^n$ be an isometric immersion defined as

$$(x_1, \ldots, x_p) \mapsto (f_1(x_1, \ldots, x_p), \ldots, f_n(x_1, \ldots, x_p)).$$

Then $\eta = |\mathrm{Min}(J(f)_x)| dx_1 \wedge \ldots \wedge dx_p$ is a nonull $p$-form on $\overline{M}$, $\forall x \in M$; where $J(f)_x$ is a Jacobian of $f$ in $x$, $\mathrm{Min}(J(f)_x)$ is the $p$-matrix (minor) obtained to Jacobian with nonull determinant, and $|\mathrm{Min}(J(f)_x)|$ is its determinant.

## 3. Invariant Lightlike Kaehler Submanifolds

Let $(M^{2n}, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifolds of an indefinite Kaehler manifolds $(\overline{M}, \overline{g}, \overline{J})$. Then $M^{2n}$ is said to be an *invariant lightlike almost complex submanifolds*, if

$$\overline{J}_x(T_xM) = T_xM, \quad \forall x \in M. \tag{28}$$

If we denote $J$ the restriction of $\overline{J}$ on $TM$, then we have

$$g(JX,\ JY) = g(X,\ Y). \tag{29}$$

**Proposition 3.1.** (1) *$Rad(TM)$ and $S(TM)$ are invariant by $J$.*

(2) *$\forall x \in M$, subspaces $Rad(T_xM), S(T_xM), S(T_xM^{\perp})$, and $ltr(T_xM)$ are even dimensions.*

**Proof 3.1.** (1) Let $\xi \in Rad(TM)$, we have $\forall X \in TM$ $g(J\xi,\ X) = g(\xi,\ JX) = 0$, then $J\xi \in Rad(TM)$. Since $J^2 = -1$, then $JRad(TM) = Rad(TM)$. It is the same for $S(TM)$.

(2) Let $W \in S(TM)$, we have $g(JW, W) = -g(W, JW) = g(W, JW) = 0$, then $JW$ is orthogonal to $W$. As $S(TM)$ is invariant, then the dimension of vector space $S(T_xM)$, $\forall x \in M$ is even.

Then, we deduce dimensions of subspaces $Rad(T_xM)$, $S(T_xM^{\perp})$, and $ltr(T_xM)$, $\forall x \in M$.

**Theorem 3.1.** *Let $(M^{2n},\ g,\ J)$ be an invariant lightlike almost complex submanifolds of an indefinite Kaehler manifolds. Then $M^{2n}$ is a lightlike Kaehler manifolds. This means*

$$\nabla J = 0 \ \textit{and}\ N_J = 0,$$

*where $\nabla$ is the induced connection on $M$.*

**Proof 3.2.**

- According to the relation (1), the Gauss formula is given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,\ Y), \quad \forall X,\ Y \in TM, \tag{30}$$

where $\nabla_X Y \in TM$ and $h(X,\ Y) \in tr(TM)$. Using the equation $\overline{\nabla}\overline{J} = 0$, we obtain

$$\nabla_X(JY) - J(\nabla_X Y) + h(X,\ JY) - \overline{J}(h(X,\ Y)) = 0.$$

Then,

$$h(X, JY) - \overline{J}(h(X, Y)) = 0, \tag{31}$$

and

$$\nabla_X(JY) - J(\nabla_X Y) = 0. \tag{32}$$

Thus $\nabla J = 0$.

• With a straightforward computation and using relations (24), (31), and (32), we have $N_J = 0$.

These results are independent to the choice of screen distribution.

**Definition 3.1.** Let $f : M \to (\overline{M}, \overline{J})$ be an isometric immersion of a lightlike almost complex submanifold into an indefinite Kaehler manifolds. Then $f$ is a holomorphic isometric immersion, if $f_\star J = Jf_\star$ on $M$, where $\overline{J}_{\setminus M} = J$.

**Proposition 3.2.** *If $f$ is a holomorphic isometric immersion, then $TM^{\perp_w} = TM^{\perp_g}$ and $Rad^w(TM) = Rad^g(TM)$.*

**Proof 3.3** (Proof of Theorem 1.1)**.** $f$ is holomorphic, i.e., $f_\star J = Jf_\star$ on $M$. Recall $f^\star \overline{g} = g$ and $\forall X, Y \in TM$, $\overline{g}(X, JY) = w(X, Y)$.

$$\begin{aligned} f^\star\overline{g}(X, JY) &= \overline{g}(f_\star X, f_\star JY) \\ &= \overline{g}(f_\star X, Jf_\star Y) \\ &= f^\star w(X, Y) \\ &= g(X, JY). \end{aligned}$$

Since $g$ is degenerated.

Moreover, we show that $f^\star w(X, Y) = g(X, JY)$ is a Kaehler form on $M$ and we have $df^\star w = f^\star dw = 0$. Thus $df^\star w(X, Y, Z) = (\nabla_X g)(Y, JZ) - (\nabla_Y g)(X, JZ) + (\nabla_Z g)(X, JY) = 0$. Then, we deduce the result.

**Proof 3.4** (Proof of Corollary 1.2)**.** Let $\{\xi_1, \dots, \xi_k, J\xi_1, \dots, J\xi_k, X_{k+1}, \dots, X_n, JX_{k+1}, \dots, JX_n\}$ be a local system coordinate of $T_xM$, $\forall x \in M$. Recall

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \forall\ X, Y \in \Gamma(TM), \tag{33}$$

where $h^l(X, Y) = L(h(X, Y))$ and $h^s(X, Y) = S(h(X, Y))$.

According to Theorem 1.1 and relation (22), $h^l = 0$. Moreover, $h(\xi_i, \xi_i) + h(J\xi_i, J\xi_i) = 0, \forall i \in \{1, \dots, k\}$ and $h(X_i, X_i) + h(JX_i, JX_i) = 0$, $\forall i \in \{k+1, \dots, n\}$. Then,

$$\sum_{i=1}^{k}(h^s(\xi, \xi) + h^s(J\xi, J\xi)) + \sum_{i=k+1}^{n}(h^s(X_i, X_i) + h^s(JX_i, JX_i)) = 0.$$

**Proof 3.5** (Proof of Corollary 1.1)**.** $f^\star w(X, Y) = g(X, Y)$ (is a Kaehler form) $\forall X, Y \in TM$ and $df^\star w = 0$. According to Theorem 1.1 and relation (22), $\nabla$ is metric, then $h^l = 0$.

**Corollary 3.1.** *A totally lightlike almost complex submanifold of a holomorphic isometric immersion is Lagrangian submanifold.*

**Corollary 3.2.** *Let* $f : (M^{2n}, g) \to (\overline{M}^{2n}, \overline{g}, \overline{J}, w)$ *be a holomorphic isometric immersion of a lightlike isotrope almost complex submanifold into an indefinite Kaehler manifolds. Then* $f^\star w = 0$, *but* $(M^{2n}, g)$ *is not Lagrangian.*

In this result sure, $f^\star w = 0$ on $M$, but $m \neq 2n$.

In the following section, we will endow the type of this submanifold a symplectic form, which we call an associate symplectic form.

**Example 3.1.** Let $M$ be a totally lightlike surface of symplectic manifolds $(\mathbb{R}^4_2, \overline{g}, \overline{J}, W)$, where

$$\bar{g}((x_1, x_2, y_1, y_2); (u_1, u_2, v_1, v_2)) = -(x_1u_1, x_2u_2) + (y_1v_1 + y_2v_2), \quad (34)$$

$$\bar{J}(x_1, x_2, y_1, y_2) = (-x_2, x_1, -y_2, y_1), \quad (35)$$

$$W((x_1, x_2, y_1, y_2); (u_1, u_2, v_1, v_2)) = (x_1u_2 - x_2u_1) - (v_2y_1 - v_1y_2). \quad (36)$$

Suppose the surface $M$ is given by equations

$$x_3 = F(x_1, x_2); \quad x_4 = G(x_1x_2),$$

with $(F'_{x_1})^2 + (G'_{x_1})^2 = 1$; $(F'_{x_2})^2 + (G'_{x_2})^2 = 1$, and $F'_{x_1}F'_{x_2} + G'_{x_1}G'_{x_2} = 0$. Then $M$ is totally lightlike surface. We have

$$TM^{\perp} = TM = Span\{\xi_1 = \frac{\partial}{\partial x_1} + F'_{x_1}\frac{\partial}{\partial x_3} + G'_{x_1}\frac{\partial}{\partial x_4};$$

$$\xi_2 = \frac{\partial}{\partial x_2} + F'_{x_2}\frac{\partial}{\partial x_3} + G'_{x_2}\frac{\partial}{\partial x_4}\}.$$

Hence Jacobian matrix of functions $F$ and $G$ is orthogonal, thus, there exists a smooth function $u$ such that $F'_{x_1} = G'_{x_2} = \cos u$; $F'_{x_2} = -G'_{x_1} = \sin u$. A straightforward calculation shows that $\bar{J}(TM) = TM$.

The immersion isometric defines by

$$f : M \to \mathbb{R}^4_2,$$

$$(a, b) \mapsto (a, b, F(a, b), G(a, b))$$

is holomorphic and $f^{\star}W = 0$.

**Example 3.2.** Let $M$ be a coisotropic lightlike submanifold of symplectic manifolds $(\mathbb{R}^6_2, \bar{g}, \bar{J}, W)$, where

$$\bar{g}((x_1, x_2, y_1, y_2, z_1, z_2); (u_1, u_2, v_1, v_2, \nu_1, \nu_2))$$

$$= -(x_1u_1, x_2u_2) + (y_1v_1 + y_2v_2 + z_1\nu_1 + z_2\nu_2),$$

$$\bar{J}(x_1, x_2, y_1, y_2, z_1, z_2) = (-x_2, x_1, -y_2, y_1, -z_2, z_1),$$

$$W((x_1, x_2, y_1, y_2, z_1, z_2); (u_1, u_2, v_1, v_2, \nu_1, \nu_2))$$

$$= -(-x_1u_2 + x_2u_1) + (-v_2y_1 + v_1y_2 - z_1\nu_2 + z_2\nu_1).$$

The submanifold $M = \{(x_1, x_2, y_1, y_2, z_1, z_2) \in \mathbb{R}^6_2;\ x_1 = y_1;\ x_2 = y_2\}$ is defined by the following equations:

$$f : M \to \mathbb{R}^6_2,$$

$$(\theta^1, \theta^2, \theta^3, \theta^4) \mapsto (x_1, x_2, y_1, y_2, z_1, z_2),$$

such that $\begin{cases} x_1 = \theta^1, \\ x_2 = \theta^2, \\ y_1 = \theta^1, \\ y_2 = \theta^2, \\ z_1 = \theta^3, \\ z_2 = \theta^4, \end{cases}$

$f$ is a natural injection, then $f$ is holomorphic. $TM$ is spanned by

$$U_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}; \quad U_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}; \quad U_3 = \frac{\partial}{\partial z_1}; \quad U_4 = \frac{\partial}{\partial z_2},$$

$Rad(TM) = TM^{\perp} = span\{U_1, U_2\}$. We have $JU_1 = U_2$ and $JU_3 = U_4$. Hence $J(TM) = TM$, $M$ is invariant with $J$. Moreover, $f^{\star}W = -d\theta^3 \wedge d\theta^4$.

**Example 3.3.** Let $M$ be a coisotropic lightlike submanifold of symplectic manifolds $(\mathbb{R}^8_4, \overline{g}, \overline{J}, W)$, where $\overline{g}, \overline{J}$, and $W$ are defined as in relations (34), (35), and (36). The submanifold $M = \{(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) \in \mathbb{R}^8_4;\ x_1 = x_3;\ y_1 = y_3;\ x_2 = x_3 + x_4;\ y_2 = y_3 + y_4\}$ is an invariant holomorphic 2-lightlike submanifold. $TM$ is spanned by

$$U_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}; \quad U_2 = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3};$$

$$U_3 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}; \quad U_4 = \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_4},$$

$Rad(TM) = span\{U_3, U_4\}$. We have $JU_1 = U_2$ and $JU_3 = U_4$. Hence $J(TM) = TM$, $M$ is invariant with $J$.

Moreover, $f^{\star}W = d\theta^1 \wedge d\theta^3 + d\theta^1 \wedge d\theta^4 + d\theta^2 \wedge d\theta^3$.

**Proposition 3.3.** *In considering the relation* (14), *the different geometry objects verify the following relations*:

$$A_{\overline{J}W}X = JA_W X; \quad D^l(X, \overline{J}W) = \overline{J}D^l(X, W); \tag{37}$$

$$\nabla^s_X(\overline{J}W) = \overline{J}\nabla^s_X(W); \quad A_{\overline{J}N}X = JA_N X; \tag{38}$$

$$D^s(X, \overline{J}N) = \overline{J}D^s(X, N); \quad \nabla^l_X(\overline{J}N) = \overline{J}\nabla^l_X(N). \tag{39}$$

With a straightforward computation, we prove this proposition.

## 4. An Associate Symplectic Form on 2-Lightlike Kaehler Submanifolds

### 4.1. Symplectic normalization

Let $(M^{2n}, S(TM), S(TM^{\perp}), g, J)$ be a 2-lightlike Kaehler submanifolds of an indefinite symplectic manifolds $(\overline{M}^{2m}, \overline{g}, \overline{J}, w)$. Then $f^{\star}w$ is not symplectic because it is degenerated on the radical distribution. To endow $M$, a symplectic form from $f^{\star}w$, we will consider pairs of the normalizing null vectors $(N, \xi)$ and $(\overline{J}N, \overline{J}\xi)$ such that $d(\theta \wedge J\theta) = 0$, where $\theta = \overline{g}(N, .)$ and $J\theta = \overline{g}(\overline{J}N, .)$.

**Lemma 4.1.** $d(\theta \wedge J\theta) = 0$ *iff* $\overline{\nabla}(N) = 0$.

We will prove this lemma with easy computation.

Several authors considered the normalization problem in various ways, but this normalization is in Bejancu and Duggal [5] approach. In which the induced geometric objects with respect not only to the screen distribution, but also to the choice of pair of the normalizing null vectors.

This normalization is like the Duggal and Bejancu one in which, we impose null vectors to be parallel.

**Proposition 4.1.** *Let* $(N, \xi)$, $(JN, J\xi)$ *be pair of normalizing null vectors as in previous subsection and make a change with* $(\widetilde{N}, \widetilde{\xi})$, $(J\widetilde{N}, J\widetilde{\xi})$, *where* $\widetilde{N} = \phi N + \zeta + W$ *and* $\zeta \in \Gamma(TM)$, $\phi \in C(M, \mathbb{R})^*$ *and* $W \in S\ (TM^{\perp})$. *Then,*

(1) $\widetilde{\xi} = \dfrac{1}{\phi}\xi.$

(2) $2\phi\theta(\zeta) + |\zeta|^2 + |W|^2 = 0.$

(3) $\widetilde{\theta}_{/M} = \phi\theta + \overline{g}(\zeta, .).$

(4) $\begin{cases} h^s(\zeta, .) & = \nabla^s W, \\ D^l(., W) & = (\nabla\phi)N, \\ \nabla\zeta & = A_{w.} \end{cases}$

(5) *Let* $\widetilde{\eta} = f^{\star}w + \widetilde{\theta} \wedge J\widetilde{\theta}$, *then* $\widetilde{\eta}^n = \phi^2\eta^n$.

**Proof 4.1.** (1) We have

$$\begin{aligned} \overline{g}(\widetilde{N}, \xi) &= \overline{g}(\phi N + \zeta + W, \xi) \\ &= \overline{g}(\phi N, \xi) \\ &= \phi. \end{aligned}$$

Then $\widetilde{\xi} = \dfrac{1}{\phi}\xi.$

(2) We have $\overline{g}(\widetilde{N}, \widetilde{N}) = 0$, then deduce the result.

(3) As $\theta = \overline{g}(N, .)$ and the fact that $\forall X \in TM$, $\overline{g}(W, X) = 0$. Then, we obtain the result.

(4) Comes from relations (14) and (15) to (19).

(5) With easy computation, we deduce the result.

**4.2. Associate symplectic form**

**Proof 4.2** (Proof of Theorem 1.2)**.**

• As $g_{\setminus S(TM)}$ and $\theta \wedge J\theta$ are, respectively, non degenerate on $S(TM)$ and $rad(TM)$ and $J$ is a bundle isomorphism, then the 2-form $\eta$ has maximal rank. It is easy to compute that $\eta^n$ is a volume element according to [7].

• The rest it is to show that $d\eta = 0.$ Then,

$$d\eta(X, Y, Z) = (\nabla_X g)(Y, JZ) - (\nabla_Y g)(X, JZ) + (\nabla_Z g)(X, JY) + d(\theta \wedge J\theta)(X, Y, Z).$$

We conclude with Theorem 1.1 and Lemma 4.1.

Thus, we have a symplectic form pseudo-compatible with degenerate metric *g* on *M*.

**Corollary 4.1.** *The* 2*-isotrope or* 2*-totally lightlike invariant almost complex submanifold of indefinite Kaehler manifold admits an associate symplectic form*

$$\eta = \theta \wedge J\theta.$$

**Remark 4.1.** (1) Suppose $\xi,$ $\theta,$ and $\eta$ are defined on $U$ and $\xi^*,$ $\theta^*,$ and $\eta^*$ are defined on $U^*.$ Then, there exists a function $\varphi$ on $U \cap U^*$ such that $\varphi^\star\eta^* = \eta,$ where the Jacobian of $\varphi,$ $[\varphi_\star]$ is given by

$$[\varphi_\star] = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in O(2n-2),$ $B \in \mathcal{M}(2n-2, 2),$ $C \in \mathcal{M}(2, 2n-2),$ and

$$D = \frac{1}{a^2 + b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} a, b \in \mathbb{R}^*.$$

(2) In another way to endow a lightlike submanifold of an indefinite Kaehler manifold, an associate symplectic form, we can at first associate to lightlike submanifold an associate nodegenerate metric and use the

Kaehler form of the above metric as symplectic form. As an example of the 2-lightlike submanifold, an associate metric express as $\widetilde{g} = g + \theta \otimes \theta + J\theta \otimes J\theta$.

(3) We can extend all this result on the $r$-lightlike submanifold of an indefinite Kaehler manifold. But, the problem is to characterize the diffeosymplectic of changing local coordinate.

### 4.3. Hamiltonian fields

Let $(M, g, S(TM), S(TM^{\perp}), \eta)$ be a lightlike symplectic submanifold. Let note $\mathfrak{L}(M, \eta) = \{X \in TM, d(i_X \eta) = 0\}$ the set of vector fields, which preserve the symplectic form $\eta$ and $harm(M, \eta) = \{X \in \mathfrak{L}(M, \eta), i_X \eta = df\}$, the set of Hamiltonian fields. Recall that $harm(M, \eta) = \mathfrak{L}(M, \eta)$, if the cohomology group $H^1(M)$ is trivial.

**Proof 4.3** (Proof of Theorem 1.3)**.** From Lemma 4.1, we have $d\theta = 0$ and $dJ\theta = 0$, or $i_{\xi}\eta = J\theta$ and $i_{J\xi}\eta = \theta$. Then, we deduce the result.

**Example 4.1.** Let use the previous Example 3.1. We define an associate symplectic form as $\eta = \theta \wedge \bar{J}\theta$, where $\theta = \frac{1}{2}\{dx_1 + F'_{x_1}dx_3 + G'_{x_1}dx_4\}$ and $\bar{J}\theta = \frac{1}{2}\{dx_2 - G'_{x_1}dx_3 + F'_{x_1}dx_4\}$. An easy computation, gives $\xi_1, \xi_2 \in harm(M, \eta)$. Then $i_{\xi_1}\eta = dx_2$ and $i_{\xi_2}\eta = dx_1$. Let defined Poisson bracket as

$$\forall f,\ g \in \mathcal{C}^{\infty},\ \{f,\ g\} = \eta(df^{\#},\ dg^{\#}),$$

where $df = (\xi.f)\theta + (J\xi.f)J\theta$ and $df^{\#}$ is such that $df(df^{\#}) = 1$.

## 5. Lightlike Symplectic Reduction

Let $(M^{2n}, g, S(TM), S(TM^{\perp}))$ be a lightlike holomorphic submanifolds of an indefinite Kaehler manifolds $(\overline{M}, \bar{g}, \bar{J}, w)$.

**Corollary 5.1.** *The subbundle $Rad(TM)$ is involutive.*

**Proof 5.1.** It comes from Theorem 1.1 and Theorem 2.7 of [5, pg. 162].

Let $f : M \to \overline{M}$ be a holomorphic isometric immersion. Let $\mathfrak{F}$ be the radical foliation defined on $M$ by $Rad(TM)$ (Frobenius theorem). The 2-form $w_M = f^{\star}w$ is constant along the leaves of $\mathfrak{F}$. Indeed, if $\xi$ is a tangent vector to the leaves of $\mathfrak{F}$, i.e., $\xi$ is section of $\in Rad(TM)$. Then,

$$L_\xi w_M = di_\xi w_M + i_\xi dw_M.$$

But $dw_M = 0$ and $i_\xi w_M = 0$, since $\xi \in TM^{\perp}$. Therefore, $L_\xi w_M = i_\xi w_M = 0$. This means that $w_M$ is a $\mathfrak{F}$-basic form.

**Proof 5.2** (Proof of Theorem 1.4)**.** Suppose $P$ is a smooth manifold and $\pi : M \to P$ is the natural projection, there exists a 2-form $w_P$ on such that $\pi^{\star} w_P = w_M$. Clearly, $TP \cong TM / Rad(TM)$ and $w_P(\overline{x}) = w_M(x)$ for an $x \in M$ projecting $\overline{x} \in P$. Hence $w_P$ is a symplectic form on $P$ [14]. Thus, the screen symplectic manifold $(P, w_P)$ is said to be obtained by reduction from $(\overline{M}, w)$.

For more details on space of leaves, we can see the part A of [1].

**Corollary 5.2.** *The proper an invariant r-lightlike holomorphic submanifolds of an indefinite Kaehler manifold admits a symplectic reduction.*

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## References


[1] J. A. Alvarez Lopez and E. Garcia-Rio, Differential Geometry, Proceedings of VIII Inter. Colloquiun, World Scientific.

[2] A. Banyaga, Symplectic geometry and related structures, Cubo Math. J. 6(1) (2004), 123-138.

[3] M. Barros and A. Romero, Indefinite Kaehler manifolds, Math. Ann. 261 (1982), 55-62.

[4] A. Cannas da Silva, Lecture Note on Symplectic Geometry, Springer-Verlag, Berlin, Heidelberg, 2001.

[5] K. L. Duggal and A. Bejancu, Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications, Mathematics and its Applications, Kluwer Acad. Publishers, Dordrecht, 1996.

[6] K. L. Duggal and A. Bejancu, Spacelike geometry of CR-structures.

[7] J. Ezin, M. Hassirou and J. Tossa, Divergence theorem for symmetric (0, 2)-tensor fields on semi-Riemannian manifold with boundary, Kodai Math. J. 30(1) (2007), 41-54.

[8] E. Fazilet, Degenerate Hermitian manifolds, Math. Phys. Anal. Geom. 8 (2005), 361-387.

[9] T. Fukami and S. Ishihara, Almost Hermitian structure on $S^6$, Tohoku Math. J. 7 (1955), 151-156.

[10] D. N. Kupeli, Degenerate manifolds, Geom. Dedicata 23(3) (1987), 259-290.

[11] A. Nersessian and E. Ramos, Massive spinning particles and geometry of null curves, Phys. Lett. B 445 (1998), 123-128.

[12] J. Rosen, Embedding of various relativistic Riemannian spaces in pseudo-Riemannian spaces, Rev. Moder. Phys. 37(1) (1965), 204-214.

[13] W. Thurston, Some examples of symplectic manifolds, Proc. Amer. Math. Soc. 55 (1976), 467-468.

[14] A. Weinstein, Lectures on Symplectic Manifolds, Amer. Math. Soc. BSMS 55, Providence, RI, 1977.


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